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On the Rate of Convergence of Some Operators on Functions of Bounded Variation

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Let $L_n(f, x)$ denote the Feller operator where f is a function of bounded variation. The rates of convergence are determined by estimating $|L_n(f, x) - f(x)|$ in terms of certain bounds. These results extend and sharpen the results of Cheng [*J. Approx. Theory* **39** (1983), 259–274] for Bernstein polynomials. Several classical operators are discussed as examples.

1. INTRODUCTION

For $f(x)$ on $[0, 1]$, let $B_n(f, x)$ be the Bernstein polynomial defined by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}, \qquad n = 1, 2, ... \tag{1.1}$$

Herzog and Hill [4] have shown that, if x is a point of discontinuity of the first kind, then

$$\lim_{n \to \infty} B_n(f, x) = (f(x^+) + f(x^-))/2. \tag{1.2}$$

Consequently, if f is a function of bounded variation on $[0, 1]$ ($f \in BV[0, 1]$) then (1.2) holds for all $x \in (0, 1)$. Cheng [2] estimated the rate of convergence of $B_n(f, x)$ for $f \in BV[0, 1]$ by proving the following result.

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THEOREM 1. *Let $f \in BV[0,1]$ and let $V_{[a,b]}(f)$ represent the total variation of f on $[a,b]$. Then for every $x \in (0,1)$ and $n \geqslant K_1(x)$ we have*

$$|B_n(f,x) - \bar{f}(x)| \leqslant \frac{K_2(x)}{n} \sum_{k=1}^{n} V_{I_k^*}(g_x) + \frac{K_3(x)}{n^{1/6}} \tilde{f}(x), \tag{1.3}$$

where $I_k^* = [x - x/\sqrt{k}, x + (1-x)/\sqrt{k}]$, $K_1(x) = (3/(x(1-x)))^8$, $K_2(x) = 3/(x(1-x))$, $K_3(x) = 18(x(1-x))^{-5/2}$, $\bar{f}(x) = (f(x^+) + f(x^-))/2$, $\tilde{f}(x) = |f(x^+) - f(x^-)|$, *and*

$$g_x(t) = \begin{cases} f(t) - f(x^+) & \text{if } x < t \leqslant 1 \\ 0 & \text{if } t = x \\ f(t) - f(x^-) & \text{if } 0 \leqslant t < x. \end{cases}$$

Although $K_2(x)$ could be improved, the first term on the right of (1.3) is asymptotically sharp as shown by Cheng [2]. However, the second term on the right of (1.3) can be improved considerably.

In this paper (1.3) is extended in three ways. We provide a modified form of (1.3) which will (i) hold for all n, (ii) be asymptotically sharp, (iii) and hold for a more general class of operators including the classical operators such as Bernstein, Szász, Baskakov, Gamma, and Weierstrass operators.

We shall consider the following operator due to Feller [3]. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with finite variance such that $E(X_1) = x \in I \subseteq R = (-\infty, \infty)$, $\text{Var}(X_1) = \sigma^2(x) > 0$. Set $S_n = X_1 + X_2 + \cdots + X_n$. For a function f, define the approximation operator as

$$L_n(f,x) = E\{f(S_n/n)\} = \int_{-\infty}^{\infty} f(t/n)\, dF_{n,x}(t), \tag{1.4}$$

where $F_{n,x}(t)$ is the distribution function (df) of S_n and $|f|$ is $F_{n,x}$-integrable. Khan [5, 6] provided the properties of $L_n(f,x)$ for $f \in C(I)$. In this paper we consider $f \in BV(I)$.

Section 2 provided the main result. Some special cases of the main result are listed in Section 3.

2. MAIN RESULTS

Throughout it is assumed that $\sigma^2(x) > 0$, otherwise one gets a trivial degenerate case. Also for the rate of convergence of $L_n(f,x)$ to $\bar{f}(x)$ it is assumed that $E|X_1|^3 < \infty$. If $f \in BV[a,b]$ where $-\infty < a < b < \infty$ then one can extend f over $(-\infty, \infty)$ by $f(t) = f(a)$ for $t < a$ and $f(t) = f(b)$ for $t > b$. Therefore the extended $f \in BV(-\infty, \infty)$. Throughout we shall use the notation f for both f and its extended version interchangeably. Furthermore, unless otherwise stated, it will be assumed that f is normalized. The main result can be stated as follows:

THEOREM 2. *Let $f \in BV(-\infty, \infty)$. Then for every $x \in (-\infty, \infty)$ and all $n = 1, 2, \ldots$ for the Feller operator* (1.4) *we have*

$$|L_n(f, x) - \bar{f}(x)| \leqslant \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{Q(x)}{n^{1/2}} \tilde{f}(x), \tag{2.1}$$

where $I_k = [x - 1/\sqrt{k}, x + 1/\sqrt{k}]$, $k = 1, 2, \ldots, n$, $I_0 = (-\infty, \infty)$, $P(x) = 2\sigma^2(x) + 1$, $Q(x) = 2E|X_1 - x|^3/\sigma^3(x)$, $\bar{f}(x) = (f(x^+) + f(x^-))/2$, $\tilde{f}(x) = |f(x^+) - f(x^-)|$, *and*

$$g_x(t) = \begin{cases} f(t) - f(x^+) & \text{if } t > x \\ 0 & \text{if } t = x \\ f(t) - f(x^-) & \text{if } t < x. \end{cases} \tag{2.2}$$

Furthermore, (2.1) *is asymptotically sharp when $f \in BV(-\infty, \infty)$ and $F_{1,x}(t)$ is either absolutely continuous with respect to the Lebesgue measure or is a lattice point distribution and x is a lattice point.*

Let $M_n = \{t: P(S_n/n \leqslant t) = P(S_n/n \geqslant t)\}$.

COROLLARY 1. *In Theorem 2 if $x \in M_n$ then*

$$|L_n(f, x) - \bar{f}(x)| \leqslant \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x) \tag{2.3}$$

regardless of the size of the saltus of f at x.

In particular, if $F_{1,x}(t)$ has a symmetric (about x) density then (2.3) holds. This result is useful for the Weierstrass operator.

COROLLARY 2. *In Theorem 2 if f is normalized everywhere except at $t = x$ then the following modification of* (2.1) *can be made*;

$$|L_n(f, x) - \bar{f}(x)| \leqslant \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{Q(x)}{n^{1/2}} (\tilde{f}(x) + 1.2\,|\hat{f}(x)|),$$

where $\hat{f}(x) = f(x) - \bar{f}(x)$.

COROLLARY 3. *In Theorem 2 if f is normalized everywhere except at*

$t = x$, $x \in M_n$, and $F_{1,x}(t)$ is absolutely continuous with respect to the Lebesgue measure then

$$|L_n(f, x) - \bar{f}(x)| \leqslant \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x).$$

The proofs of the theorem and the corollaries will be based on the following theorem and lemmas. Theorem 3 is the well-known Berry–Esseen bound for the classical central limit theorem of probability theory. Its proof and further discussion can be found in Loève [7, p. 300], Feller [3, p. 515], and Shiryayev [8, p. 342].

THEOREM 3. *If $E|X_1|^3 < \infty$ then there exists a numerical constant τ, $(2\pi)^{-1/2} \leqslant \tau < 0.8$, such that for all $n = 1, 2, ...$ and all t*

$$|F^*_{n,x}(t) - G^*(t)| \leqslant \frac{\tau E|X_1 - x|^3}{\sqrt{n}\, \sigma^3(x)}, \tag{2.4}$$

where $F^*_{n,x}(t)$ is the df of $\sqrt{n}(S_n/n - x)/\sigma(x)$ and $G^*(t)$ is the df of the standard normal random variable, i.e.,

$$G^*(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-u^2/2)\, du. \tag{2.5}$$

LEMMA 1. *Let $sgn_x(t)$ be defined as*

$$\operatorname{sgn}_x(t) = \begin{cases} 1 & \text{if } t > x \\ 0 & \text{if } t = x \\ -1 & \text{if } t < x. \end{cases} \tag{2.6}$$

Then

$$|L_n(\operatorname{sgn}_x, x)| \leqslant \begin{cases} 0 & \text{if } x \in M_n \\ R(x)/\sqrt{n} & \text{if } x \notin M_n, \end{cases}$$

where $R(x) = 5\tau E|X_1 - x|^3/\sigma^3(x)$.

Proof. Clearly $L_n(\operatorname{sgn}_x, x) = P(S_n/n > x) - P(S_n/n < x) = 0$ if $x \in M_n$. If $x \notin M_n$, we have

$$L_n(\operatorname{sgn}_x, x) = 1 - 2F^*_{n,x}(0) + P(S_n/n = x).$$

Thus,

$$|L_n(\operatorname{sgn}_x, x)| \leqslant 2|F^*_{n,x}(0) - G^*(0)| + |F^*_{n,x}(0) - F^*_{n,x}(0^-)| \tag{2.7}$$

Also,

$$|F_{n,x}^*(0)-F_{n,x}^*(0^-)| \leqslant |F_{n,x}^*(0)-G^*(0)|+|F_{n,x}^*(0^-)-G^*(0)| \quad (2.8)$$

and

$$|F_{n,x}^*(0^-)-G^*(0)| \leqslant \begin{cases} |F_{n,x}^*(\varepsilon_n)-G^*(0)| & \text{if } F_{n,x}^*(0^-)>\frac{1}{2} \\ |F_{n,x}^*(-\varepsilon_n)-G^*(0)| & \text{if } F_{n,x}^*(0^-)<\frac{1}{2}, \end{cases}$$

where $\varepsilon_n=(2\pi)^{1/2}\tau n^{-1/2}E|X_1-x|^3\,\sigma^{-3}(x)$. Now,

$$|F_{n,x}^*(\pm\varepsilon_n)-G^*(0)| \leqslant |F_{n,x}^*(\pm\varepsilon_n)-G^*(\pm\varepsilon_n)|+|G^*(\pm\varepsilon_n)-G^*(0)| \quad (2.9)$$

and

$$|G^*(\pm\varepsilon_n)-G^*(0)| \leqslant \frac{\varepsilon_n}{\sqrt{2\pi}}. \quad (2.10)$$

By Lemma 1 (2.7) through (2.10) the result follows.

The following lemma is useful for the proof of Corollary 2. In this case, f is not necessarily normalized at $t=x$.

LEMMA 2. *Let $\delta_x(t)$ be defined as follows*:

$$\delta_x(t)=\begin{cases} 1 & \textit{if } t=x \\ 0 & \textit{if } t\neq x. \end{cases}$$

Then

$$|L_n(\delta_x, x)| \leqslant \begin{cases} 0 & \textit{if } x \textit{ is a point of continuity of } \bar{F}_{n,x}(t) \\ 3R(x)/5\sqrt{n} & \textit{otherwise,} \end{cases}$$

where $R(x)$ is provided in Lemma 1 *and $\bar{F}_{n,x}(t)$ is the df of S_n/n.*

Proof. If x is a point of continuity of $\bar{F}_{n,x}(t)$, then

$$L_n(\delta_x, x)=P(S_n/n=x)=0.$$

Otherwise,

$$L_n(\delta_x, x)=F_{n,x}^*(0)-F_{n,x}^*(0^-).$$

Now, by (2.8), (2.9), and (2.10) the lemma follows.

LEMMA 3. *Let* $g_x(t) \in BV(-\infty, \infty)$ *such that* $g_x(t)=0$ *at* $t=x$. *Then for all* $n=1, 2, ...$

$$|L_n(g_x, x)| \leqslant \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x), \tag{2.11}$$

where $P(x)=2\sigma^2(x)+1$, $I_k=[x-k^{-1/2}, x+k^{-1/2}]$, $k=1, 2, ..., n$, *and* $I_0=(-\infty, \infty)$.

Proof. The proof of this lemma is based on the method of Bojanic and Vuilleumier [1] (see Cheng [2] also). We will denote $V_{[a,b]}(g_x)$ by $V_{[a,b]}$ for short. Consider the following three integrals separately,

$$L_n(g_x, x)=\left(\int_{-\infty}^{\alpha}+\int_{\alpha}^{\beta}+\int_{\beta}^{\infty}\right) g_x(t)\, d\bar{F}_{n,x}(t),$$

where $\bar{F}_{n,x}(t)$ is the *df* of S_n/n, $\alpha=x-1/\sqrt{n}$, and $\beta=x+1/\sqrt{n}$. Now,

$$\left|\int_{\alpha}^{\beta} g_x(t)\, d\bar{F}_{n,x}(t)\right| \leqslant \int_{\alpha}^{\beta} |g_x(t)-g_x(x)|\, d\bar{F}_{n,x}(t) \leqslant V_{I_n}. \tag{2.12}$$

By integration by parts we have

$$\int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t)=g_x(\alpha^+)\,\bar{F}_{n,x}(\alpha)+\int_{-\infty}^{\alpha} \hat{\bar{F}}_{n,x}(t)\, d(-g_x(t)),$$

where $\hat{\bar{F}}_{n,x}(t)$ is the normalized form of $\bar{F}_{n,x}(t)$. Now, $t \leqslant \alpha < x$, $|d(-g_x(t))| \leqslant d_t(-V_{[t,x]})$. Also $\hat{\bar{F}}_{n,x}(t) \leqslant \bar{F}_{n,x}(t) \leqslant n^{-1}\sigma^2(x)(t-x)^{-2}$ for all $t \leqslant \alpha$ by Chebyshev's inequality. Therefore,

$$\left|\int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t)\right| \leqslant V_{[\alpha,x]}\sigma^2(x)+\frac{\sigma^2(x)}{n}\int_{-\infty}^{\alpha} \frac{1}{(x-t)^2}\, d_t(-V_{[t,x]}). \tag{2.13}$$

Now

$$\int_{-\infty}^{\alpha} \frac{1}{(x-t)^2}\, d_t(-V_{[t,x]})=-nV_{[\alpha^+,x]}+\int_{-\infty}^{\alpha} \frac{2}{(x-t)^3}\, V_{[t,x]}\, dt, \tag{2.14}$$

where $V_{[\alpha^+,x]}=\lim_{\varepsilon \downarrow 0} V_{[\alpha+\varepsilon,x]}$. Let $u=(x-t)^{-2}$

$$\int_{-\infty}^{\alpha} \frac{2}{(x-t)^3}\, V_{[t,x]}\, dt=\int_{0}^{n} V_{[x-u^{-1/2},x]}\, du \leqslant \sum_{k=0}^{n} V_{[x-k^{-1/2},x]}, \tag{2.15}$$

where for $k=0$ we take $V_{[x-k^{-1/2},x]}=V_{(-\infty,x]}$ Hence, by (2.13), (2.14), and (2.15) we have

$$\left|\int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t)\right| \leqslant V_{[\alpha,x]}\,\sigma^2(x) + \frac{\sigma^2(x)}{n}\sum_{k=0}^{n} V_{[x-k^{-1/2},x]}$$

$$\left|\int_{-\infty}^{\alpha} g_x(t)\, d\bar{F}_{n,x}(t)\right| \leqslant \frac{2\sigma^2(x)}{n}\sum_{k=0}^{n} V_{[x-k^{-1/2},x]}. \tag{2.16}$$

On the other hand,

$$\int_{\beta}^{\infty} g_x(t)\, d\bar{F}_{n,x}(t) = \int_{\beta}^{\infty} g_x(t)\, d(-\bar{S}_{n,x}(t)),$$

where $\bar{S}_{n,x}(t) = 1 - \bar{F}_{n,x}(t) = P(S_n/n > t)$ is the left continuous, nonincreasing survival function. Again, integrating by parts, applying Chebyshev's inequality, and repeating (2.13), (2.14), (2.15), and (2.16) we have

$$\left|\int_{\beta}^{\infty} g_x(t)\, d\bar{F}_{n,x}(t)\right| \leqslant \frac{2\sigma^2(x)}{n}\sum_{k=0}^{n} V_{[x,x+k^{-1/2}]}, \tag{2.17}$$

where for $k=0$ we take $V_{[x,x+k^{-1/2}]} = V_{[x,\infty)}$. Combining (2.12), (2.16), and (2.17) we get

$$|L_n(g_x, x)| \leqslant \frac{2\sigma^2(x)}{n}\sum_{k=0}^{n} V_{I_k} + V_{I_n} \leqslant \frac{2\sigma^2(x)+1}{n}\sum_{k=0}^{n} V_{I_k}.$$

This completes the proof of Lemma 3.

Proof of Theorem 2. First note that for all t

$$f(t) = g_x(t) + \bar{f}(x) + \tfrac{1}{2}(f(x^+) - f(x^-))\,\mathrm{sgn}_x(t),$$

where $g_x(t)$ and $\mathrm{sgn}_x(t)$ are given by (2.2) and (2.6), respectively. By linearity of the operator, the triangular inequality, and Lemmas 1 and 3 the bound follows by taking $Q(x) = R(x)/2$. To show the sharpness of the asymptotic rate of convergence, consider the functions

$$p(t) = \begin{cases} |t-x| & \text{if } t \in [x-\varepsilon, x+\varepsilon] \\ \varepsilon & \text{if } t \notin [x-\varepsilon, x+\varepsilon], \end{cases} \tag{2.18}$$

where $\varepsilon > 0$, and

$$q(t) = \begin{cases} 1 & \text{if } t > x \\ \frac{1}{2} & \text{if } t = x \\ 0 & \text{if } t < x. \end{cases} \tag{2.19}$$

It is easy to show (see Khan [6] for a more general result) that

$$\lim_{n\to\infty} n^{1/2}E|S_n/n-x| = \sqrt{2/\pi}\,\sigma(x).$$

By Chebyshev's inequality,

$$\lambda_n = \sqrt{n}\,\varepsilon P\{|S_n/n-x|>\varepsilon\}\to 0 \qquad \text{as} \quad n\to\infty,$$

and by the Cauchy–Schwarz and Chebyshev inequalities

$$\sqrt{n}\int_{|t-x|>\varepsilon}|t-x|\,d\bar{F}_{n,x}(t)\to 0 \qquad \text{as} \quad n\to\infty.$$

Therefore,

$$\sqrt{n}\,|L_n(p,x)-p(x)| = \lambda_n + \sqrt{n}\int_{|t-x|\leqslant\varepsilon}|t-x|\,d\bar{F}_{n,x}(t)\to\sqrt{2/\pi}\,\sigma(x)$$

as $n\to\infty$. On the other hand, $p(x^+)=p(x^-)=\bar{p}(x)=p(x)=0$, $g_x(t)=p(t)$ and for sufficiently large n,

$$\sum_{k=0}^{n} V_{I_k}(g_x) = V_{(-\infty,\infty)}(g_x) + \sum_{k=1}^{[1/\varepsilon^2]}\left(V_{I_k}(g_x)-\frac{2}{\sqrt{k}}\right)+\sum_{k=1}^{n}\frac{2}{\sqrt{k}}.$$

Therefore for sufficiently large n,

$$\sum_{k=0}^{n} V_{I_k}(g_x)\leqslant\mu+\nu\sqrt{n},$$

where μ and ν are constants. Therefore the first term on the right side of (2.1) is asymptotically sharp for $p(t)$. Similarly consider $q(t)$. Now, $q(x)=\bar{q}(x)$, $g_x(t)\equiv 0$ and

$$|L_n(q,x)-\bar{q}(x)| = \tfrac{1}{2}|L_n(\operatorname{sgn}_x,x)|$$

or

$$|L_n(q,x)-\bar{q}(x)| = |F^*_{n,x}(0)-G^*(0)-(\tfrac{1}{2})P(S_n/n=x)|. \tag{2.20}$$

Now it is well known (cf. Feller [3, p. 512]) that if $F_{1,x}(t)$ is absolutely continuous with respect to the Lebesgue measure then

$$F^*_{n,x}(t)-G^*(t)-\frac{E(X_1-x)^3}{6\sigma^3(x)\sqrt{n}}(1-t^2)\,g^*(t)=o\left(\frac{1}{\sqrt{n}}\right) \tag{2.21}$$

uniformly in t, where $g^*(t)$ is the first derivative of $G^*(t)$. Therefore, for the case when $F_{1,x}(t)$ is absolutely continuous with respect to the Lebesgue measure, $P(S_n/n = x) = 0$, and the asymptotic sharpness of the second term in (2.1) follows. Furthermore, if $F_{1,x}(t)$ is a lattice distribution, then (2.20) reduces to

$$|L_n(q, x) - \bar{q}(x)| = |\hat{F}_{n,x}(0) - G^*(0)|,$$

where $\hat{F}_{n,x}(0) = (F^*_{n,x}(0) + F^*_{n,x}(0^-))/2$. Again it is well known (Feller [3, p. 514) that (2.21) holds for the lattice distribution where $F^*_{n,x}(t)$ is replaced by $\hat{F}_{n,x}(t)$ and t is a lattice point. Consequently, the asymptotic sharpness of the bound follows. This completes the proof of Theorem 2.

The proof of Corollary 1 is obvious. To prove Corollary 2, note that for all t

$$f(t) = g_x(t) + \bar{f}(x) + \tfrac{1}{2}(f(x^+) - f(x^-)) \operatorname{sgn}_x(t) + \hat{f}(x)\, \delta_x(t),$$

where $g_x(t)$ and $\operatorname{sgn}_x(t)$ are given by (2.2) and (2.6), respectively, and $\delta_x(t)$ is provided in Lemma 2 and $\hat{f}(x) = f(x) - \bar{f}(x)$. Now the proof of the corollary is straightforward by using Lemma 2. The proof of Corollary 3 is also straightforward.

3. Special Cases

In the following we provide a few examples for the classical operators. The emphasis is on the explicit values of $P(x)$ and $Q(x)$ in each case.

3.1. *Bernstein Operator*

Let $P(X_1 = 1) = x = 1 - P(X_1 = 0)$, $x \in (0, 1)$. Then

$$L_n(f, x) = B_n(f, x).$$

Now, $\sigma^2(x) = x(1-x)$ and $E|X_1 - x|^3 = \sigma^2(x)(2x^2 - 2x + 1)$. Therefore,

$$|B_n(f, x) - \bar{f}(x)| \leqslant \frac{2x(1-x)+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2(2x^2 - 2x + 1)}{\sqrt{nx(1-x)}} \tilde{f}(x) \tag{3.1}$$

We may remark in passing that $I_k = [x - k^{-1/2},\ x + k^{-1/2}]$ could be replaced by $I_k^* = [x - xk^{-1/2}, x + (1-x)\, k^{-1/2}]$ and that (3.1) would be modified as

$$|B_n(f, x) - \bar{f}(x)| \leqslant \frac{K_2(x)}{n} \sum_{k=1}^{n} V_{I_k^*}(g_x) + \frac{2(2x^2 - 2x + 1)}{\sqrt{nx(1-x)}} \tilde{f}(x), \tag{3.2}$$

where $K_2(x)$ is given in Theorem 1 and $n \geqslant 1$. This result improves the bound obtained by Cheng (Theorem 1).

3.2. *Szász Operator*

Let $P(X_1 = j) = e^{-x}x^j(j!)^{-1}$, $j = 0, 1, \ldots$. Then we obtain the Szász operator $S_n(f, x)$

$$L_n(f, x) = S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} f(k/n) \frac{(nx)^k}{k!},$$

where $f(t)$ is defined over $[0, \infty)$. Now, $\sigma^2(x) = x$ and

$$E|X_1 - x|^3 = x + 2e^{-x} \sum_{k=0}^{[x]} (x-k)^3 \frac{x^k}{k!}.$$

Therefore, if $f \in BV[0, \infty)$ then for all $x \in (0, \infty)$ we have

$$|S_n(f, x) - \bar{f}(x)| \leqslant \frac{2x+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2E|X_1 - x|^3}{x^{3/2}\sqrt{n}} \tilde{f}(x). \tag{3.3}$$

One could use a simple bound for $E|X_1 - x|^3 \leqslant 8x^3 + 6x^2 + x$ to have

$$|S_n(f, x) - \bar{f}(x)| \leqslant \frac{2x+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2(8x^2 + 6x + 1)}{\sqrt{nx}} \tilde{f}(x).$$

3.3. *Baskakov Operator*

Now let $P(X_1 = j) = 1/(1+x)(x/(1+x))^j$, $j = 0, 1, \ldots, x \in (0, \infty)$. In this case, (1.4) defines the Baskakov operator $B_n^*(f, x)$ as follows:

$$B_n^*(f, x) = (1+x)^{-n} \sum_{k=0}^{\infty} f(k/n) \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k. \tag{3.4}$$

Now $\sigma^2(x) = x(1+x)$ and

$$E|X_1 - x|^3 = 2x^3 + 3x^2 + x + \frac{2}{1+x} \sum_{j=0}^{[x]} (x-j)^3 \left(\frac{x}{1+x}\right)^j$$

$$|B_n^*(f, x) - \bar{f}(x)| \leqslant \frac{2x(1+x)+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2E|X_1 - x|^3}{(x(1+x))^{3/2}\sqrt{n}} \tilde{f}(x). \tag{3.5}$$

Again one could use the bound $E|X_1-x|^3 \leqslant 16x^3+9x^2+x$ to obtain

$$|B_n^*(f, x)-\bar{f}(x)| \leqslant \frac{2x(1+x)+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x)+\frac{2(16x^2+9x+1)}{(1+x)^{3/2}\sqrt{nx}} \tilde{f}(x).$$

3.4. *Gamma Operator*

Let X_1 have probability density over $(0, \infty)$

$$f_{X_1}(y)=\begin{cases}(1/x)\, e^{-y/x} & \text{if } y>0 \\ 0 & \text{if } y \leqslant 0.\end{cases}$$

Then (1.4) defines the Gamma operator

$$G_n(f, x)=\frac{x^{-n}}{(n-1)!} \int_0^{\infty} f(y/n)\, y^{n-1} e^{-y/x}\, dy.$$

Now $\sigma^2(x)=x^2$ and $E|X_1-x|^3=2x^3(6/e-1)$. Hence, for any $f \in BV[0, \infty)$ and $x \in (0, \infty)$ we have

$$|G_n(f, x)-\bar{f}(x)| \leqslant \frac{2x^2+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x)+\frac{4(6/e-1)}{\sqrt{n}} \tilde{f}(x). \tag{3.6}$$

3.5. *Weierstrass Operator*

Let X_1 have probability density defined over $(-\infty, \infty)$ by $g^*(y-x)$ where $g^*(y)$ is the derivative of $G^*(y)$ given in (2.5). Then $L_n(f, x)$ reduces to the Weierstrass operator

$$W_n(f, x)=\sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} f(x+u) \exp(-(nu^2)/2)\, du.$$

Now, $\sigma^2(x)=1$ and by Corollary 1 we have

$$|W_n(f, x)-\bar{f}(x)| \leqslant \frac{3}{n} \sum_{k=0}^{n} V_{I_k}(g_x)$$

for $f \in BV(-\infty, \infty)$ and $x \in(-\infty, \infty)$ regardless of the size of the saltus of f at x.

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